

LONGEVITY/MORTALITY RISK MODELING AND SECURITIES PRICING

Yinglu Deng
Patrick L. Brockett
Richard D. MacMinn

ABSTRACT

Securitizing longevity/mortality risk can transfer longevity/mortality risk to capital markets. Modeling and forecasting mortality rate is key to pricing mortality-linked securities. Catastrophic mortality and longevity jumps occur in historical data and have an important impact on security pricing. This article introduces a stochastic diffusion model with a double-exponential jump diffusion process that captures both asymmetric rate jumps up and down and also cohort effect in mortality trends. The model exhibits calibration advantages and mathematical tractability while better fitting the data. The model provides a closed-form pricing solution for J.P. Morgan's q-forward contract usable as a building block for hedging.

INTRODUCTION

The terms "longevity risk" and "mortality risk" have attracted the attention of insurance companies, annuity providers, pension funds, and investment banks. The definitions are as follows (Coughlan et al., 2007). Longevity risk describes the risk that an individual or group will live a longer life than expected (their mortality rate will be lower than expected), while mortality risk describes the risk that an individual or group will live a shorter life than expected (their mortality rate will be higher than expected). Clearly, life insurers are interested in mortality risk while annuity providers, defined benefit plans and social insurance programs such as Social Security, are interested in longevity risk.

In the last several decades, life expectancy for populations in the developed world has, on average, been increasing by approximately 1.2 months every year. Globally, life expectancy at birth has increased by 4.5 months per year on average over the

Yinglu Deng is at Department of Finance, School of Economics and Management, Tsinghua University. Patrick L. Brockett is at Department of Information, Risk, and Operations Management, Red McCombs School of Business, The University of Texas at Austin. Richard D. MacMinn is at Katie School of Insurance, College of Business, Illinois State University. The authors can be contacted via e-mail: dengyinglu@gmail.com, brockett@mail.utexas.edu, and richard.macminn@ilstu.edu, respectively.

second half of the 20th century (Gutterman et al., 2002). Substantial improvements in longevity at older ages during the 20th century have challenged longevity risk management in pension funds that originally reserved using what would now be considered incorrectly diminished mortality rates. Further, documenting this issue, a 2006 study of the companies in the UK's FTSE100 index found that many companies had based their estimates of pension liabilities on mortality tables that underestimated expected lifetimes by not recognizing improving longevity, and that recognizing this underestimation of expected lifetimes would cause the aggregate deficit in pension reserves, to more than double from £46 billion to £100 billion (Jardine Lloyd Thompson, 2006). In 2010 alone, improved life expectancy added £5 billion to corporate pension obligations in the United Kingdom (Reuters News Service, 2010). In the United States, according to Watson Wyatt, the implementation of the recently established Internal Revenue Service's (IRS) mortality assumptions for the level of pension contributions needed for adequate reserving will increase pension liabilities by 5–10 percent (Halonnen, 2007). Similarly, Mercer Human Resource Consulting has calculated that the use of up-to-date mortality tables would increase the cost of providing a pension to a male born in 1950 by 8 percent (Smithson, 2006).

On the other side of the mortality/longevity risk problem, many life insurers globally have become concerned about their potential exposure to catastrophic mortality risks, such as flu. During the 1918 flu, more than 675,000 excess deaths from the flu occurred between September 1918 and April 1919 in the United States alone, causing a huge jump in the mortality rate time series. More recently, H5N1 avian influenza occurred in Hong Kong in 1997, and H1N1 occurred globally in 2009. According to the American Council of Life Insurers (ACLI), in 2009, the reserves for U.S. life insurance policies were around \$3.8 trillion (ACLI, 2010); however, there is a possibility that a major pandemic event could trigger insolvency in the life insurance industry, and this situation worsens if catastrophic mortality events coincide with financial downturns (Toole, 2007).

The interaction of the insurance industry and the capital markets (Cummins, 2005) has provided a vehicle for mitigating the above mentioned mortality/longevity risk, namely, through the financial securitization (cf. MacMinn, Brockett, and Blake, 2006). Securitization provides an approach to off-loading nondiversifiable risk from the insurer or pension balance sheet and transferring it to the capital market. This may provide an attractive alternative to reinsurance. According to Cummins and Trainer (2009, p. 475) "... the traditional reinsurance model begins to break down when risks are correlated, add significantly to the reinsurer asymmetry risk, and are large relative to the reinsurer's equity capital. The cost of capital is also increased by informational asymmetries between reinsurers and capital market and by agency costs and other market frictional costs. Under these conditions, the price of reinsurance may be prohibitively high, and the supply of coverage may be restricted." Longevity risk for pension plans or annuity providers is an example of such correlated risk (mortality improvements affect most individuals). In these circumstances securitization can help address the inefficiencies in the reinsurance market such as correlation between risks, counterparty credit risk in the case of large or catastrophic risk (cf. Cummins and Trainer, 2009). Moreover, the capital markets are also significantly larger in capacity than the reinsurance markets, so insolvency risk is reduced, and since the securitized

insurance products tend to be uncorrelated with other assets in the economy, they can be attractive to investors wishing to diversify their own risks. Securitization also enhances the risk capacity of the insurance industry, as illustrated by the catastrophe (CAT) mortality bond and other similar derivatives, whose payment depends on the underlying loss indices and the catastrophic mortality event. On the insurance policyholder side, many investment banks have recently been involved in the life-settlement securitization. Investment banks purchased hundreds of thousands of life insurance policies and repackaged them into bonds, then sold bonds to investors such as pension funds. The expected rate of return to an investor in such a bond depends on the projected life expectancy of the members in the pool of life insurance policies (Modu, 2009).

Pricing of the CAT mortality bonds or life-settlement securities depends on the estimation and forecast of mortality rates or life expectancy, which are considerations involving mortality risk and longevity risk. The estimation and forecast of life expectancy or mortality rate also plays a crucial role in longevity/mortality risk management for pension funds or insurers. In this article, we propose a stochastic model, based on the Brownian motion process, plus an asymmetric jump diffusion process for the estimation and forecasting of mortality rates and life expectancy.

Since the longevity and mortality jumps are critical sources of risk of which pension funds and insurers should be cognizant (Zanjani, 2002), these jumps should be incorporated in the modeling and securitization process (Cox, Lin, and Pedersen 2010). The mortality jumps (such as the 1918 flu) have a short-term intensified effect, while the longevity jumps (caused by the pharmaceutical or medical innovation) have long-term gentle effect. The different frequency and intensity of the mortality jumps and the longevity jumps explain the distribution skewness of the mortality time-series increment, which is important but not considered in previous mortality rate models. Considering the asymmetric jump phenomenon of the mortality time series, our model adopts a compound Poisson double-exponential jump diffusion (DEJD) process to capture the longevity jumps and the mortality jumps, respectively.

Very few studies address modeling mortality jumps for securitization. Biffis (2005) and Bauer, Borger, and Russ (2010) apply affine jump diffusion processes to model the force of mortality in a continuous-time framework. Our model incorporates the cohort effect, which captures the mortality time series and adjusts it to fit different age groups. The model with cohort effects capture this feature of the historical mortality time series more accurately. Chen and Cox (2009) model the jump process with a compound Poisson normal jump diffusion process. Our contribution is applying the DEJD with different longevity and mortality jumps. This captures the distribution skewness of the mortality time-series increment and provides a better fit to the empirical data. Cox, Lin, and Pedersen (2010) propose a model to accommodate both longevity jumps and mortality jumps. Our model has fewer parameters and more concise specification, and can be easily parameterized and applied to securitization.

Our model incorporates the advantages of the Lee–Carter mortality framework. It describes the mortality time series and considers the cohort effect, making age-specific

adjustment for different age groups. This adjustment for different age groups is critical for the model, since mortality improvement and extreme positive or negative events (such as influenza pandemic) have different intensity levels in different age groups. In this way, the Lee–Carter model appropriately describes the three-dimensional surface of the mortality rate with respect to the time horizon and the age horizon. The Lee–Carter framework has been extended by Brouhns, Denuit, and Vermunt (2002), Renshaw and Haberman (2003), Denuit, Devolder, and Goderniaux (2007), Li and Chan (2007), Chen and Cox (2009), and Lin and Cox (2005, 2008).

We test our model with historical data and make model fitness comparisons with previous models. The results clearly illustrate an improvement in the fit to the data. In our application, we use the first CAT mortality bond, the Swiss Re Catastrophe Mortality Bond (2003), to calculate the implied risk premium with two changes of measure approaches. We implement our model to price the q -forward longevity risk derivative contract as an example illustrating the benefit of our model for providing a closed-form pricing solution for this standard structured mortality-linked security. The q -forward derivative was proposed by J.P. Morgan based on the LifeMetrics index. It has the potential to be a future standardized contract and help to create a liquid market. In addition to the q -forward, our model can provide closed-form pricing solutions to other mortality-linked securities with cash flows that are linear functions of the mortality rate for each period.

Our article is organized as follows. In the “Data Description” section, we introduce the historical mortality rate and describe the mortality rate age-specific features and the mortality time series. In the “Model” section, we describe the model requirement and specification which includes a baseline Brownian motion process and a compound Poisson DEJD process. In the “Numerical Solution” section, the numerical method is applied to calibrate the model and compare the fitness of the model with previous ones. In the “Example: q -Forward Pricing” section, we use the Swiss Re catastrophe mortality bond as the benchmark product to calculate the implied market price of risk. Based on the implied market price of risk and our model, we price the standard market product q -forward as a pricing example. We conclude and discuss the article in the “Conclusion.”

DATA DESCRIPTION

The historical data come from HIST290 National Center for Health Statistics, the same data used in Chen and Cox (2009). We use the same data in order to facilitate the model fitness comparison in the “Numerical Solution” section. The data list death rates per 100,000 population for selected causes of death. Death rates are tabulated for age group (>1), (1–4), (5–14), (15–24), then every 10 years, to (75–84), and (>85), including both sex and race categories. Selected causes for death include major conditions such as heart disease, cancer, and stroke. Figure 1 shows the mortality rate for different age groups and years. Figure 2 shows the comparison of the mortality rate for several sample age groups, including relatively older groups and younger groups.

We can observe clearly the two properties of the mortality rate trend from Figure 1 and Figure 2. In Figure 1, the downward trend indicates that the mortality rate follows a

FIGURE 1
1900–2004 Mortality Rates

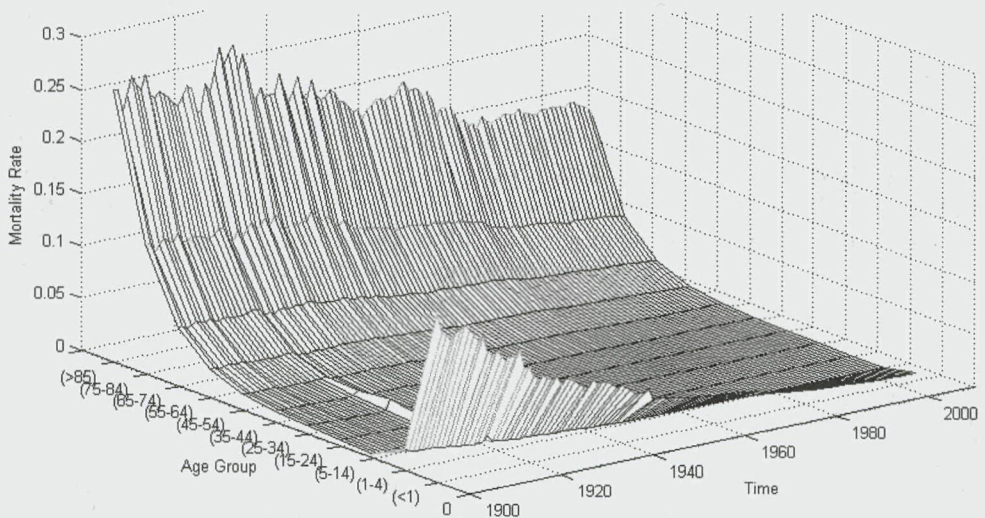
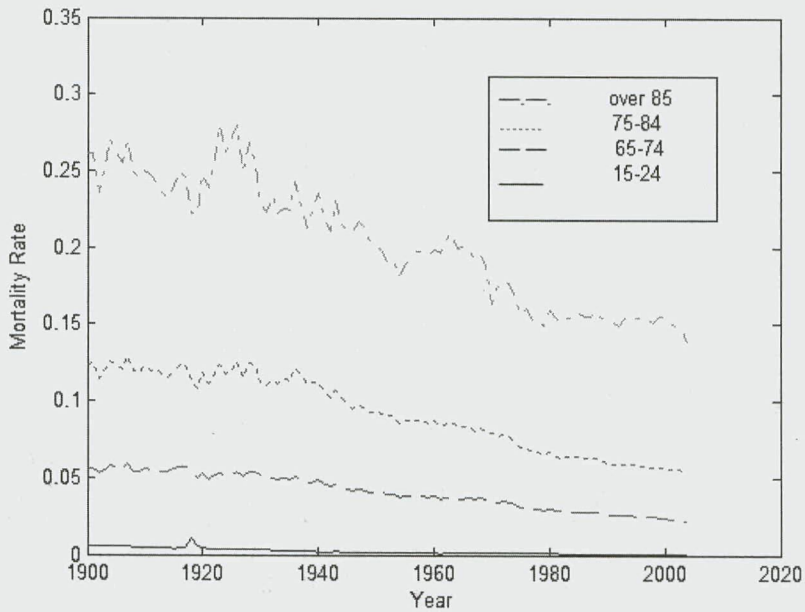


FIGURE 2
Comparison of the Age Group Mortality Rates



decreasing trend during 1900–2004, at all ages. For example, in the over 85 age group, the mortality rate decreases from 0.26 to 0.14, while in the younger age groups, such as 15–24, the mortality rate decreases from 0.006 to 0.0008. The decreasing trend shows the improvement of the life times or longevity in all age groups.

In Figure 2, the change in the mortality rate in the older age groups is more significant with a steeper downward trend than that in the younger age groups. For example, the mortality rate decreases 0.12 in the age group over 85. During the same time period, the mortality rate decreases only 0.0052 in the age group 15–24. The comparison of the steepness of the mortality surface shows that the improvement in longevity among the older age group is more significant than that in the younger age group. We can observe the variability and dynamics in the mortality trends. Accordingly, we shall use a stochastic model to capture the dynamics features.

MODEL

Model Framework and Requirement

The basic requirement of the mortality model is to capture the features described by the data. Various mortality rate models have been provided by previous research. The majority of models are based on the Lee–Carter one-factor model (Lee and Carter, 1992). These models extend the Lee–Carter model to a two-factor model (Blake, Cairns, and Dowd, 2006a), or incorporate a stochastic process in the model (Dahl, 2004), or introduce the possibility of a jump process in the stochastic process to accommodate extreme outliers in the mortality time series (Cox, Lin, and Wang, 2006; Chen and Cox, 2008).

In the Lee–Carter framework, the mortality rate $\mu_{x,t}$ denotes the mortality rate of the group whose age is x during the year t . It is decomposed into age-specific parameters a_x , b_x , and mortality time-series k_t :

$$\ln(\mu_{x,t}) = a_x + b_x k_t + e_{x,t}, \quad (1)$$

$$\mu_{x,t} = \exp(a_x + b_x k_t + e_{x,t}). \quad (2)$$

Here a_x represents the age group shift effect, and $\exp(a_x)$ is the general shape across the age of the mortality schedule, and b_x represents the age group's reaction effect to the mortality time-series k_t . In other words, the b_x profile tells us which group of mortality rates decline rapidly and which group of mortality rates decline slowly in response to changes in k_t , and $e_{x,t}$ captures the age group's residual effect not reflected in the model. Lee and Carter (1992) suggest estimating the parameters in their model using a two-stage singular value decomposition (SVD) based on historical data for $\mu_{x,t}$ to estimate the age-specific parameters a_x and b_x , and to generate the mortality time-series k_t .

To implement the SVD procedure, first, we need to normalize using a condition that the sum of the k_t is equal to 0 and the sum of the b_x is equal to 1. Then a_x must equal

TABLE 1Fitted Value for Age-Specific Parameters a_x and b_x During 1900–2004

Age Group	a_x	b_x
<1	-3.4087	0.1455
1–4	-6.2254	0.1960
5–14	-7.1976	0.1492
15–24	-6.2957	0.0994
25–34	-5.9923	0.1044
35–44	-5.4819	0.0855
45–54	-4.7799	0.0608
55–64	-4.0137	0.0468
65–74	-3.2347	0.0426
75–84	-2.4196	0.0409
>85	-1.6119	0.0290

the average over time of $\ln(\mu_{x,t})$:

$$a_x = \frac{1}{T} \sum_{t=1}^T \ln(\mu_{x,t}). \quad (3)$$

Furthermore, k_t must (or almost) equal the sum over age of $(\ln(\mu_{x,t}) - a_x)$, since the sum of b_x has been chosen to be unity. This is not an exact relation, however, since the error terms will not in general sum to 0 for a given age. Then, each b_x can be found by regressing, without a constant term, $(\ln(\mu_{x,t}) - a_x)$ on k_t separately for each age group x .

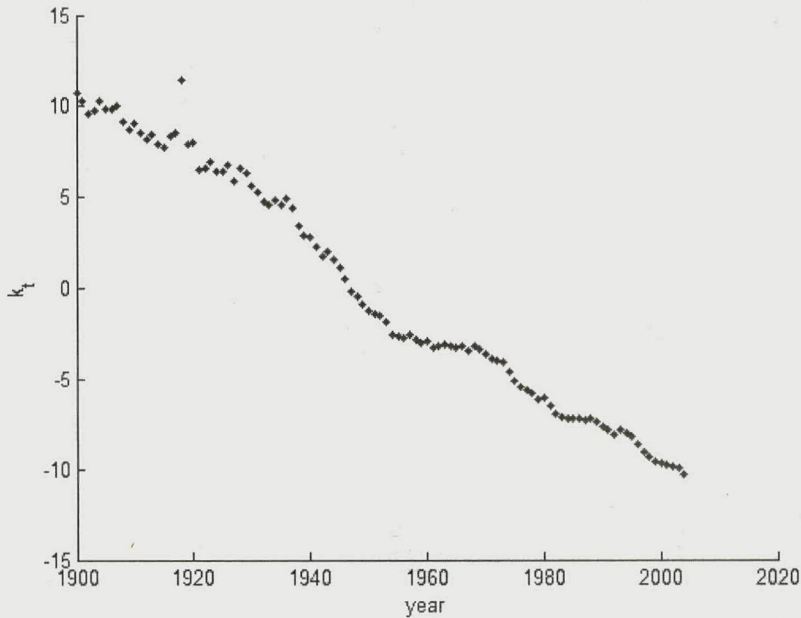
In the second stage, we reestimate the mortality time-series k_t iteratively, given the estimation of a_x and b_x in the first step. This enables the actual sum of death at time t (left-hand side) to equal the implied sum of deaths at time t (right-hand side).

$$D_t = \sum_x (P_{x,t} \exp(a_x + b_x k_t)), \quad (4)$$

where D_t is the actual sum of deaths at time t , and $P_{x,t}$ is the population in age group x at time t .

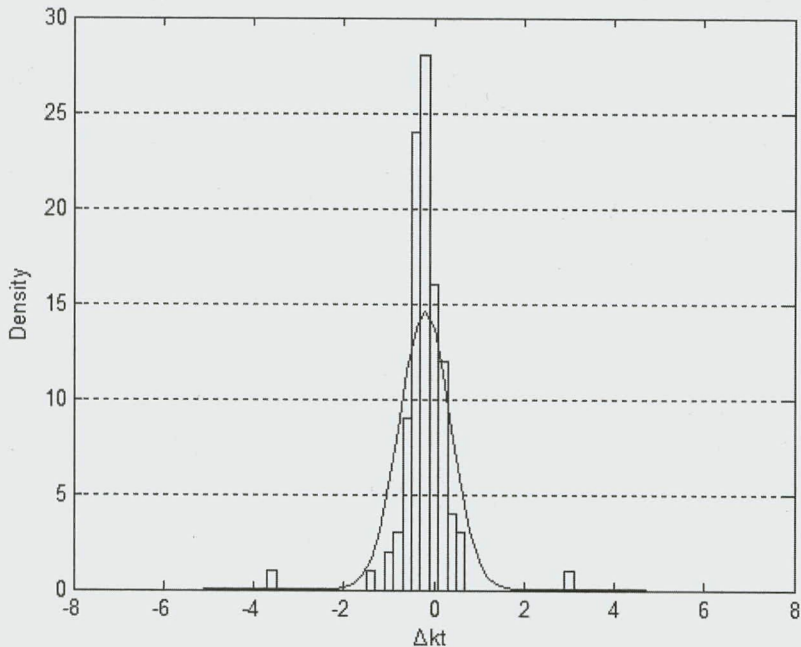
Implementing the SVD two-stage procedure with data on historical U.S. mortality rates during 1900–2004, we get the fitted a_x , b_x , in Table 1, and the mortality time-series k_t in Figure 3. The decreasing trend of mortality time-series k_t shows the improvement of mortality along the time as described previously. Figure 3 also shows the big jump in 1918 (which was caused by the flu pandemic) and other jumps around 1920, 1943, etc.

Now, we need a model to capture the features of the shape, the trend and the jumps of the mortality time-series k_t . First, the model should incorporate a stochastic term in the description of the k_t time series, as this has proved to be better than using the

FIGURE 3The Mortality Time-Series k_t 

model without the stochastic process (Dahl and Muller, 2004). Second, as shown in Figure 3, k_t includes both positive and negative values. Since geometric Brownian motion will not generate a negative value from the positive starting value, it does not fit the k_t process by itself. However, Brownian motion can be selected to fit the k_t time series. Third, we can observe from Figure 3 that the jumps are transient, not permanent. For example, the sudden increase of mortality rate in 1918, caused by the flu, falls back to the normal condition in the second year.

Beyond the three points on the model specification listed above, the jump phenomenon in mortality rates is two directional. In Figure 3, the positive jumps (the sudden increases in the mortality time series) occur with higher severity, while the negative jumps (the suddenly decreases in the mortality time series) are of lower severity and higher frequency. Coughlan et al. (2007) also show that there is a significant negative autocorrelation in mortality rates, so increases in mortality are often followed by increases in longevity. However, there is a definite downward trend in mortality with the jumps being asymmetric and of unequal frequency. Hence, the model that involves a jump process with a symmetric normal distribution for the size of the jump (Chen and Cox, 2009) may not adequately capture the asymmetry of the mortality jump phenomenon. From biological and demographic perspectives, the positive jumps (mortality jumps) can be explained by sudden catastrophes (e.g., earthquake, hurricane) or critical diseases, such as the extreme positive jump caused by flu in 1918. These positive jumps can be transitory (1918 flu) or more lasting (AIDS, antibiotic resistant strains of tuberculosis, etc.). The negative jumps (longevity jumps) are associated with multiple biological and health improvement causes. According

FIGURE 4Comparison of Actual Δk_t Distribution and Normal Distribution

to Johnson, Bengtson, and Coleman (2005), increases in survival currently reflect a shift in the causes of death from infections to chronic degenerative diseases. Hence, the improvements in mortality due to health or biological reasons affecting chronic diseases appear more frequently and are of a more moderate size over time. The health improvements due to improved medical treatment, for example, may be frequent in occurrence and small in impact, but they have gradually but significantly changed life expectancy. Illustrative of this, since 1960 longevity has increased 1–1.5 percent per year due mainly to a 65 percent reduction in cardiovascular events in the age cohort over age 45 (Iacovino, 2009). Again the survival jump effect could be lasting or transitory (e.g., a new drug which loses effectiveness). A large longevity jump could occur in the future if an effective treatment of coronary heart disease and cancer were found, as these two causes of death combined constitute more than half of all deaths among people over the age of 40 (Johnson, Bengtson, and Coleman, 2005, p. 109). Ultimately, once the jump (positive or negative) occurs, any lasting effect can be subsequently captured in the Gaussian process component of the model. Because the negative jumps are of low severity and high frequency, they are difficult to discern in the figure except possibly in a more negatively inclined series.

The descriptive statistics of $\Delta k_t = k_{t+1} - k_t$ show leptokurtic features. The skewness of Δk_t equals to -0.451 . In other words, the Δk_t distribution is skewed to the left and has a higher peak and two heavier tails than those of a normal distribution, which we can also observe in Figure 4.

In Figure 4, the histogram represents the distribution of actual Δk_t , and as can be seen, Δk_t cannot be fitted well by a normal distribution. Hence, the Brownian motion process alone cannot be used to describe the mortality time-series k_t .

To incorporate the leptokurtic feature of the Δk_t distribution, the analysis here incorporates a DEJD model to capture both the positive jumps and negative jumps of the k_t process. Compared to the model of Cox–Lin–Pedersen (2010), which also captures the positive jumps and negative jumps (the size of the jumps is normally distributed so there is symmetry in the distribution in Cox and Chen), our model has a concise specification and an easy approach for calibration. What is more, unlike Cox et al. (2008), our model has a closed-form solution for the forecast of the future mortality rate, which facilitates mortality securities pricing.

The Model Specification

To capture the features of the mortality time-series k_t , and to account for the tractability and the calibration of the model, we set the model specification to describe Δk_t in the approximate continuous-time model of dk_t as given below.

The dynamics of the mortality time-series k_t is specified as

$$dk_t = \alpha dt + \sigma dW_t + d \left(\sum_{i=1}^{N(t)} (V_i - 1) \right), \quad (5)$$

where W_t is a standard Brownian motion, $N(t)$ is a Poisson process with rate λ , where λ describes the expected frequency of the jumps. The larger the λ , the more time jumps occur in the mortality time series. Here V_i is a sequence of independent identically distributed (iid) nonnegative random variables, $Y = \log(V)$ has a double-exponential distribution with the density:

$$\begin{aligned} f_Y(y) &= p\eta_1 e^{-\eta_1 y} 1_{\{y \geq 0\}} + q\eta_2 e^{\eta_2 y} 1_{\{y < 0\}}, \\ \eta_1, \eta_2 &> 0, p, q \geq 0, p + q = 1. \end{aligned} \quad (6)$$

The parameters p and q represent, respectively, the proportion of positive jumps and negative jumps among all jumps. Thus, $p\lambda$ is the expected frequency of positive jumps and $q\lambda$ is the expected frequency of negative jumps. The parameters η_1 and η_2 describe the positive jump severity and the negative jump severity separately. Thus, $Y | Y \geq 0$ is exponentially distributed with mean η_1^{-1} , while $-Y | Y \leq 0$ is exponentially distributed with mean η_2^{-1} . The larger the η_1 , the smaller the positive jumps severity. Similarly, the larger the η_2 , the smaller the negative jump in absolute value. In this way, the positive jumps and negative jumps are captured by similar distributions but with different parameters, based on the asymmetry of jumps in the mortality time-series k_t and the leptokurtic feature of dk_t .

The model specification with the double-exponential distribution has the advantage of mathematical tractability allowing a closed-form formula for the expected future mortality rate to be derived. Because of this closed-form solution, the DEJD model may

provide a useful stochastic mortality model for internal company mortality simulation as well as being useful in the capital market applications we discuss subsequently. The double-exponential distribution has also been widely implemented as a stock price jump diffusion model, for which closed-form solutions for options and other securities are available (Kou, 2004). In this article, the closed-form solution of the expected future mortality rate is presented in Equation (19).

NUMERICAL SOLUTION

Parameter Calibration

The disentangling of jumps from the diffusion components is a serious difficulty in the calibration of the underlying processes. The increments of the underlying process are supposed to be captured by the diffusion process, with a few extreme increments captured by jumps. However, the addition of the jump process may yield the wrong calibrated parameters, which leads to high frequency of jumps and small severity of jumps. The DEJD model for dk_t faces the same problem. A calibration method is needed to generate the right parameters of low frequency and large severity for jumps. Ait-Sahalia and Hansen (2004) demonstrate that maximum likelihood estimation (MLE) has advantages in disentangling jumps from diffusion. Meanwhile, the double-exponential jump diffusion is a linear process with independent increments and an explicit transition density, which fortunately satisfies the requirement of a complete specification of the transition density for using MLE. Therefore, we choose the MLE method to calibrate parameters $\{\lambda, p; \eta_1, \eta_2; \alpha, \sigma\}$ with the dk_t time series.

Let $C = \{k_0, k_1, \dots, k_T\}$ denote the mortality time series, at equally spaced times $t = 1, 2, \dots, T$. The one-period increments $r_i = \Delta k_i = k_i - k_{i-1}$ is (iid). The unconditional density of one-period increment $f(r)$ is

$$f(r) = e^{-(\lambda_u + \lambda_d)} f_{0,0}(r) + e^{-\lambda_u} \sum_{n=1}^{\infty} p(n, \lambda_d) f_{0,n}(r) \\ + e^{-\lambda_d} \sum_{m=1}^{\infty} p(m, \lambda_u) f_{m,0}(r) + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} p(n, \lambda_d) p(m, \lambda_u) f_{m,n}(r), \quad (7)$$

where $p(n, \lambda_d) = e^{-\lambda_d} \lambda_d^n / n!$, $p(m, \lambda_u) = e^{-\lambda_u} \lambda_u^m / m!$, and $f_{m,n}(r)$ is the conditional density for a one-period increment, conditional on the given number of up and down jumps (m, n) in the increment:

$$f_{0,0}(r) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(r - \mu + \frac{1}{2}\sigma^2)^2}, \quad (8)$$

$$f_{0,n}(r) = \frac{\eta_d^n}{(n-1)!\sqrt{2\pi}\sigma} \int_{-\infty}^0 (-x)^{(n-1)} e^{(\eta_d x - \frac{1}{2\sigma^2}(r-x-\mu + \frac{1}{2}\sigma^2)^2)} dx, \quad (9)$$

$$f_{m,0}(r) = \frac{\eta_u^m}{(m-1)!\sqrt{2\pi}\sigma} \int_0^\infty (x)^{(m-1)} e^{(-\eta_u x - \frac{1}{2\sigma^2}(r-x-\mu+\frac{1}{2}\sigma^2)^2)} dx, \quad (10)$$

$$f_{m,n}(r) = \frac{\eta_u^m \eta_d^n}{(m-1)!(n-1)!\sqrt{2\pi}\sigma} \int_{-\infty}^\infty \left(\int_{-\infty}^{0 \wedge t} (-x)^{n-1} (t-x)^{m-1} e^{(\eta_u + \eta_d)x} dx \right) \\ \times e^{-\eta_u t} e^{-\frac{1}{2\sigma^2}(r-x-\mu+\frac{1}{2}\sigma^2)^2} dt. \quad (11)$$

The log-likelihood given T equally spaced increment observations is

$$L(C; \lambda_u, \lambda_d; \eta_1, \eta_2; \alpha, \sigma) = \sum_{i=1}^T \ln(f(r_i)),$$

where $\lambda = \lambda_u + \lambda_d$, and $p = \frac{\lambda_u}{\lambda}$. Then we can calibrate the parameters $\{\lambda, p; \eta_1, \eta_2; \alpha, \sigma\}$.

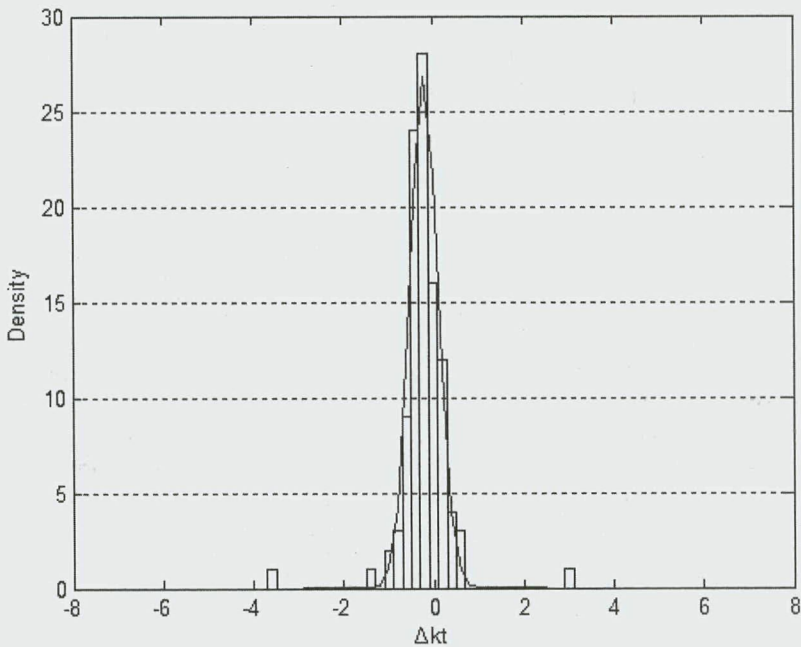
After computation, we get $\{\lambda_u, \lambda_d; \eta_1, \eta_2; \alpha, \sigma\} = \{0.029, 0.035; 0.71, 0.75; -0.20, 0.31\}$, and $\{\lambda, p; \eta_1, \eta_2; \alpha, \sigma\} = \{0.064, 0.45; 0.71, 0.75; -0.20, 0.31\}$ and maximum likelihood value $L = -49.95$. Here η_1 and η_2 describe the severity of positive jumps (mortality jumps) and negative jumps (longevity jumps), respectively. A larger η represents a smaller severity. The fact that $\eta_1 = 0.71 < 0.75 = \eta_2$ verifies that the severity of positive jumps is larger than the severity of negative jumps, consistent with that observed in Figure 3. Here λ_u and λ_d describe the frequency of positive jumps and negative jumps, respectively. A larger λ represents a larger frequency. Since $\lambda_u = 0.029 < 0.035 = \lambda_d$, this verifies that the frequency of positive jumps is smaller than the frequency of negative jumps, consistent with that observed in Figure 3.

Model Comparison

Figure 5 shows how the DEJD model fits the actual increment of mortality rate Δk_t , by comparing the distribution generated by the DEJD model calibrated to historical data and the actual distribution of Δk_t . Comparing Figure 5 with Figure 4, we observe that the DEJD extension of the Lee-Carter model approximates the distribution of increment of mortality rate Δk_t much better than the commonly used Brownian motion extension. The mean of the distribution of DEJD and the Brownian motion is the same ($\mu_{DEJD} = \mu_{BM} = -0.20$); however, the standard deviation of the distribution of DEJD ($\sigma_{DEJD} = 0.31 < \sigma_{BM} = 0.57$) is significantly less than that for the Brownian motion model. This is exactly the reason that the DEJD model is more appropriate to fit the actual distribution, which can be directly observed from the comparison of the two figures. The underlying reason is that the Lee-Carter model includes the outliers in the Brownian motion diffusion process, which causes the calibrated σ_{BM} to be larger. And the Brownian motion diffusion is appropriate to capture the normal distribution shape without a fat tail and a high peak; however, this is not the case in these data. In our DEJD model, we include the outliers in the jump diffusion part,

FIGURE 5

Comparison of Actual Δk_t Distribution and DEJD Distribution



which enables the calibrated σ_{DEJD} to be smaller than σ_{BM} and improves the fit to the data.

Next, the DEJD model is compared with both Lee–Carter Brownian motion model and the normal jump diffusion model (Chen and Cox, 2009). For model selection, we adopt the Bayesian information criterion (*BIC*) proposed by Schwarz (1978). Unlike the significance test, *BIC* allows comparison of more than two models at the same time and does not require the alternatives to be nested. *BIC* is a “conservative” criterion since it heavily penalizes over parameterization (Ramezani and Zeng, 2007).

Suppose the k th model M_k has parameter vector θ_k , where θ_k consists of n_k independent parameters to be estimated. Denote θ'_k as the MLE of θ_k . Then, *BIC* for Model M_k is defined as

$$BIC_k = -2 \ln f(C|\theta'_k, M_k) + n_k \ln(m), \quad (12)$$

where m is the number of observations in data set C and $f(C|\theta'_k, M_k)$ is the maximized likelihood function. With the *BIC* criterion, the best fitting model is the one with the smallest *BIC* value. Table 2 gives the *BIC* model fit values for the three competing models. It uses the maximum likelihood function values provided in Chen and Cox (2008). It can be observed that the DEJD model fits best.¹

¹ The DEJD model fits the real series of 104 data points better than the other models in spite of being penalized by the *BIC* criterion for having more parameters. We do get the same ranking

TABLE 2

Comparison of Model Fitness

	Number of Parameters	ln (Likelihood)	BIC
Double-exponential jump diffusion model (DEJD model)	6	-49.95	127.76
Normal jump diffusion model (Chen-Cox model)	5	-62.52	148.26
No jump diffusion model (Lee-Carter model)	2	-94.27	197.83

The underlying reasons that our DEJD model fits the data better are as follows. First, the outliers in the mortality time series cause there to be fat tails and a high peak in the increment Δk_t distribution, which rules out the normal distribution. The Lee-Carter model treats outliers the same as other points in the mortality time-series evolution process. As a result, the outliers enhance the variability of the process and cause overestimation of the standard deviation σ . Our DEJD model applies a compound Poisson DEJD process separately from the Brownian motion diffusion process. This avoids the problem of mismatching the fat tail and high peak with the normal distribution and hence provides a better fit.

Second, the Chen-Cox model applies the normal jump diffusion model, which is composed of a Brownian motion diffusion process and a normal jump diffusion process. This model also treats the outlier with a normal distribution. Actually, the positive outlier (or jump) and the negative outlier (or jump) are due to different biological and technical reasons. The positive outliers (e.g., that caused by the 1918 flu pandemic) have short-term intensified effects, while the negative outliers (caused by pharmaceutical or medical innovation) have long-term gentle effects. The different frequency and intensity of the positive outliers and the negative outliers explain the skewness of the mortality time-series increment, which is not appropriately described by the symmetric normal jump diffusion.

The BIC comparison above shows the model with asymmetric jumps fits the mortality time-series k_t better than the Lee-Carter model or the Chen-Cox model. An important feature of the data is the 1918 flu epidemic. In our view, the flu is in the data, might recur and should therefore be included in any model, especially one aimed toward financially hedging against such events in the future. In an alternative view, Lee and Carter treated it as an outlier, essentially regarding it as a one of a kind event whose inclusion would result in misleading average forecasts. Additionally, Chen and Cox (2009) and Li and Chan (2005, 2007), in order to reveal a smoother mortality trend, performed a systematic time-series outlier analysis for the mortality data in United States and Canada, and fit the adjusted outlier-free mortality series to the

of the models by log likelihood fit as by BIC, even if we exclude the 1918 flu year from the data (which we do not think should be done). However, if the 1918 flu year is excluded and the series is made artificially smoother, then using the BIC criterion, the parameter penalty dominates and the ranking is simply according to the number of parameters in the model (Lee-Carter with two parameters, then Chen-Cox with five parameters, and then DEJD model with six parameters).

Lee–Carter model. In their data, United States, 1900–2000, they find seven outliers, which occurred in years 1916, 1918, 1921, 1928, 1936, 1954, and 1975. Most of these outliers result from influenza epidemics according to their explanations, except for the data in 1954 and 1975. To explore the effect of such deletions on our model, we did another analysis where we deleted the outliers found by Li and Chan from the original mortality data and reestimated the mortality factor k_t . The parameters obtained were $\{0.01, 0.05, 0.92, 0.85, -0.23, 0.39\}$, and the $\ln$ (likelihood) value was $\{-47.62\}$ with a BIC value of 123.1063. The maximum likelihood value is similar as the result treated by Chen–Cox model and Lee–Carter model (Chen and Cox, 2010). When the jumps are treated as outliers and deleted from the trend, the parameters describing the arrival of the positive jump or negative jump converge to 0. The parameters describing the Gaussian component (drift and volatility) converge to that in the Lee–Carter model.

However, almost surely catastrophic mortality event will occur again, the question is only when, and how can vulnerable entities (life insurers, pension funds, etc.) prepare for it and mitigate their financial exposure to this risk. For the purpose of this article, the rationale for the interest in mortality and longevity risk is precisely to be able to handle such outliers as the 1918 flu pandemic. If, on the other hand, the purpose were to model mortality rates alone for aggregate smooth life table models, such as the goal of Lee–Carter, then their removal prior to parameter fitting might be justified. In our case if there were no possibility of jumps, especially large potentially bankrupting jumps in mortality or longevity rate, then there would be no interest in longevity or mortality derivatives to be introduced into the capital markets. It is precisely because of such jumps that the problem arises and hence we strongly feel that the inclusion of these outliers should be maintained. Having said that, without the 1918 outlier, as expected, the BIC improves since one is fitting a smoother series. Still, we feel that the outlier point needs to be included as a matter of principle.

Implied Market Price of Risk

Like other insurance products, such as annuities, the longevity-risk-contingent securities are priced in an incomplete capital market. Hence, the risk premium should be considered in pricing the issues, since it represents the price that pension funds or insurers are willing to pay to transfer longevity or mortality risk. In previous research articles (Blake et al., 2006; Chen and Cox, 2009), the Swiss Re mortality catastrophe bond has been used to calculate the implied mortality risk premium, given its payment structure and issue price.

The Swiss Re Mortality Catastrophe Bond

The Swiss Re insurance company issued the first mortality risk contingent securitization in December 2003. If the bond is triggered by a catastrophic evolution of death rates of a certain population, the investors incur a loss in principal and interest. The bond provides the investors higher yield as compensation for the mortality risk they take. The bond was issued through a special purpose vehicle (SPV) called Vita Capital, which enabled Swiss Re to remove extreme catastrophic risk from its balance sheet.

TABLE 3

Implied Market Prices of Risk by Risk-Neutral Approach

ζ_1	ζ_2	ζ_3
6.15	0	0
0	0.21	0
0	0.18	0.18

The bond had a maturity of 3 years, a principal of \$400 million, and a coupon rate of 135 basis points plus LIBOR rate. The mortality index, M_t , was a weighted average of mortality rates over five countries, males and females, and a range of ages. The principal was repayable in full only if the mortality index did not exceed 1.3 times the 2002 base level during any year of the bond's life. If mortality did exceed this threshold, the payment was dependent on the realized values of the mortality index. The precise payment schedules were given by the following f_t functions:

$$f_t = \begin{cases} \text{LIBOR} + \text{spread} & t = 1, \dots, T-1 \\ \text{LIBOR} + \text{spread} + \max\{0, 100\% - \sum_t L_t\} & t = T \end{cases},$$

where the function L_t specifies the amount of payment that is lost due to mortality experience, namely,

$$L_t = \begin{cases} 0\% & M_t < 1.3M_0 \\ [(M_t - 1.3M_0)/(0.2M_0)] \times 100\% & 1.3M_0 \leq M_t \leq 1.5M_0 \text{ for all } t. \\ 100\% & 1.5M_0 < M_t \end{cases} \quad (13)$$

Risk-Neutral Pricing. Risk-neutral pricing was used by Milevsky and Promislow (2001) and Blake, Cairns, and Dowd (2006a). The method is derived from financial economic theory that is applicable even in an incomplete market. If the overall market has no arbitrage, then there exists at least one risk-neutral measure Q that can be used for calculating fair prices. We apply the approach in Blake, Cairns, and Dowd (2006b), which assumes the market price of mortality risk is constant and estimate it from the Swiss Re mortality bond. As a more liquid mortality-linked securities market develops, a more accurate market price of risk can be calculated based on the adequacy of deals and data.

The payoff of the longevity derivative instrument that we consider in this article is dependent on the difference between the experienced and expected mortality (or equivalently, survival) rate. However, the mortality rate in our model for a fixed age is itself linearly dependent on the time series k_t , that we modeled using the DEJD. In an incomplete market, the risk neutral pricing will allow pricing of the derivative. Kou and Wang (2004) discuss derivative pricing using the DEJD model for security prices, and they derive the risk-neutral measure for this stochastic process. Using their results, the DEJD model in the physical measure has a risk-neutral DEJD model with parameters given below where the asterisk symbol denotes that the parameter

corresponds to the risk neutral measure:

$$dk_t = (\alpha^* - \lambda^* \gamma^*) dt + \sigma^* dW^* + d \left(\sum_{i=1}^{N^*(t)} (V_i^* - 1) \right), \quad (14)$$

$$\gamma^* = E^*[V^*] - 1 = \frac{p^* \eta_1^*}{\eta_1^* - 1} + \frac{q^* \eta_2^*}{\eta_2^* + 1} - 1. \quad (15)$$

Integrate Equation (14) to obtain

$$k_s = k_0 + \left(\alpha^* - \frac{1}{2} \sigma^{*2} - \lambda^* \gamma^* \right) s + \sigma^* W_s^* + \sum_{i=1}^{N^*(s)} Y_i^*. \quad (16)$$

According to Kou and Wang (2004) the characteristic function is

$$E^*[e^{\theta(k(s)-k_0)}] = \exp[G(\theta)s] \quad \text{or}$$

$$E^*[e^{\theta k(s)}] = \exp(\theta k_0) \exp[G(\theta)s], \quad (17)$$

where

$$G(\theta) = \theta \left(\alpha^* - \frac{1}{2} \sigma^{*2} - \lambda^* \gamma^* \right) + \frac{1}{2} \theta^2 \sigma^{*2} + \lambda^* \left(\frac{p^* \eta_1^*}{\eta_1^* - \theta} + \frac{q^* \eta_2^*}{\eta_2^* + \theta} - 1 \right). \quad (18)$$

Using b_x for θ in Equation (17), the closed-form expression for the expected future mortality rate $\mu_{x,t}$ is derived as

$$\begin{aligned} E^*[\mu_{x,t}] &= \exp(a_x) \times E^*[\exp(b_x k_t)], \\ &= \exp \left(a_x + b_x k_0 + b_x t \left(\alpha^* - \frac{1}{2} \sigma^{*2} - \lambda^* \gamma^* \right) \right. \\ &\quad \left. + \frac{1}{2} b_x^2 \sigma^{*2} t + \lambda^* t \left(\frac{p^* \eta_1^*}{\eta_1^* - b_x} + \frac{q^* \eta_2^*}{\eta_2^* + b_x} - 1 \right) \right). \end{aligned} \quad (19)$$

Equation (19) can be used to calculate the expected future mortality rate directly with parameters $\{\lambda^*, p^*; \eta_1^*, \eta_2^*; \alpha^*, \sigma^*\}$, which is much faster and more convenient than using simulation to project and average the paths of future mortality rates. This model is especially suitable for pricing the mortality-linked securities

whose cash flow each period is a linear function of the mortality rate (e.g., the q -forward). We discuss the example of q -forward pricing in the "Example: q -Forward Pricing" section.

We can derive the implied market price of risk ζ based on the actively traded mortality-linked securities on the market whose fair price is already known and then apply the same ζ to price the unknown mortality-linked securities. In the previous research, the annuity price (Cox and Lin, 2006) or the mortality bond price (Chen and Cox, 2009) was used as the known traded price. In this article, we use the Swiss Re mortality catastrophe bond to determine a known market price of mortality risk to enable us to calculate ζ , and then use this in our DEJD model to price the q -forward incorporating ζ as an implementation example of our DEJD model. For comparison purposes, we consider three possible models corresponding to the Brownian motion, the positive jump severity, and the negative jump severity, along with $\{\alpha^*, \eta_1^*, \eta_2^*\}$. Since the mortality-linked securities are priced in an existing incomplete market, the value of ζ or the risk-neutral measure Q is not unique. Practically speaking we have only one mortality-linked security to use (the Swiss Re bond) but need to calculate three market prices of risk. Accordingly, we use the process suggested by Blake, Cairns, and Dowd (2006b), to estimate the set of ζ by sequentially changing one and fixing the rest.

Consider the market prices of risk set $\zeta = \{\zeta_1, \zeta_2, \zeta_3\}$ with $\alpha^* = \alpha + \zeta_1$; $\eta_1^* = \eta_1 + \zeta_2$; and $\eta_2^* = \eta_2 + \zeta_3$. We can then estimate the components of ζ by changing only one and fixing the rest. The algorithm below is similar to the traditional procedure for calculating the market price of risk with Wang transform approach (Chen and Cox, 2009).

Step 1. Based on the known 2003 mortality time series, simulate 10,000 times the future mortality time-series $k(t)$ for 2004–2006, using the DEJD model (5) with the calibrated parameter set $\{\lambda, p; \eta_1, \eta_2; \alpha, \sigma\} = \{0.064, 0.45; 0.71, 0.75; -0.20; 0.31\}$, and the initial assumed set $\zeta = \{\zeta_1, \zeta_2, \zeta_3\} = \{0, 0, 0\}$, with the risk-neutral transform function.

Step 2. Calculate the mortality rate $\mu_{x,t}$ by the formula (2) and calculate the average μ_t based on year 2000 standard population and corresponding weights. The year 2000 standard population and corresponding weights is based on the technique notes of the NCHS report GMWK293R. The weights are 0.013818 for the <1 age group, 0.055317 for the 1–4 age group, 0.145565 for the 5–14 age group, 0.138646 for the 15–24 age group, 0.135573 for the 25–34 age group, 0.162613 for the 35–44 age group, 0.134834 for the 45–54 age group, 0.087247 for the 55–64 age group, 0.066037 for the 65–74 age group, 0.044842 for the 75–84 age group, and 0.015508 for the >85 age group.

Step 3. Calculate the expected value of the principal payment in every period T by the formula, $E_T^*[payment] = \$400,000,000 \times [\max(1 - \sum_{t=2004}^{2006} L_t, 0)]$, where L_t is given by Equation (13). The coupon payment in every period is calculated based on the par spread plus 135 basis points, the latter of which was obtained by reference to the risk premium of the Swiss Re mortality bond contract.

TABLE 4
Example of q-Forward Structure

Notional amount	\$50,000,000
Trade date	December 31, 2006
Effective date	December 31, 2006
Maturity date	December 31, 2016
Reference year	2015
Fixed rate q_{fixed}	0.8765% (or 87.65 basis points)
Fixed rate payer	XYZ investment bank
Fixed amount	Notional amount $\times$ fixed rate $\times$ 100
Reference rate	LifeMetrics index
Floating amount payer	XYZ pension or annuity provider
Floating amount	Notional amount $\times$ reference rate $\times$ 100
Settlement	Net settlement = fixed amount – Floating amount

Step 4. Iteratively adjust the market price of risk set ζ , and repeat Step 1–Step 3 until the discounted expected value of the coupon payment in 2004–2006 plus the principal payment in 2006 equals the face value of the mortality bond \$400,000,000.

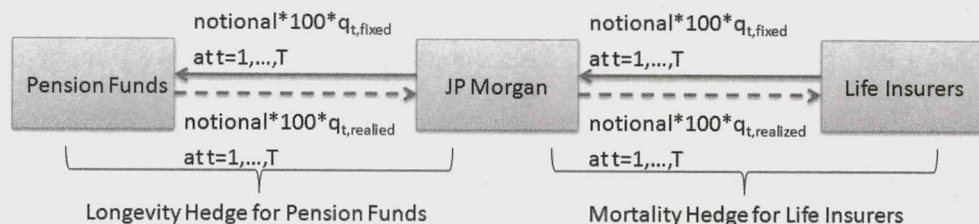
EXAMPLE: Q-FORWARD PRICING

J.P. Morgan proposed q-forward derivative contracts as simple capital markets instrument for transferring longevity and mortality risk. q-forwards enable pension funds to hedge against increasing life expectancy of plan members and life insurers to protect themselves against significant increases in the mortality of policyholders. Similar in structure to other forwards, q-forwards are securities involving the exchange of the realized mortality of a population at some future date, in return for a fixed mortality rate agreed at inception. These q-forwards can provide the basic building blocks from which many other complex mortality/longevity risk securities can be constructed. A q-forward provides a type of standardized contract that could help to create a liquid longevity risk capital market. A set of q-forwards that settle based on the LifeMetrics Mortality Index could fulfill this role. Since the counterparty who is not exposed to the longevity risk requires a risk premium to take on longevity risk, the mortality forward rates at which q-forwards transact will be below the expected, or “best estimate” mortality rates. The standard actuarial notation uses q for mortality rate, and this is how q-forward derivative is named. In our notation, q is denoted by μ .

A q-forward type contract that can be used by a life insurer to hedge mortality risk occurs when the life insurer pays the fixed mortality rate to the investment bank and the investment bank pays the realized mortality rate to the life insurer. The life insurer receives the payment (notional amount) $\times (q_{\text{realized}} - q_{\text{fixed}}) \times 100$, as shown in Table 4. When the realized mortality rate increases, the investment bank pays more (the realized rate minus the fixed rate) to the life insurer, which offsets the loss incurred by the life insurer due to the experienced increase in mortality. A q-forward type contract can also be used by a pension fund or by Social Security to hedge longevity risk. The pension fund pays the realized mortality rate to the investment bank and the investment bank pays the fixed rate to the pension fund. The pension

FIGURE 6

Longevity Risk Hedge and Mortality Risk Hedge



fund receives the payment (notional amount) $\times (q_{\text{fixed}} - q_{\text{realized}}) \times 100$. When the mortality decreases (longevity increases), the investment bank pays more (fixed rate minus realized rate) to the pension fund, which can then use these funds to cover the loss incurred by the pension fund due to the experienced increase in longevity. In this way, pension funds that are long on longevity risk can transfer the risk to investors who are willing to take on this extra risk for the increased return they receive. Similarly, life insurers who are long on mortality risk can transfer the risk to investors who want to short the mortality risk.² Of course the pension fund and the life insurer are natural counterparties, and the investment bank can serve as a financial intermediary facilitating their mutual hedging as in Figure 6.

Based on our DEJD model and the q -forward product structure above, the fixed rate can be calculated with the closed-form formula (19) directly:

$$\begin{aligned}
 E^*[\mu_t] &= \sum_x W_x \times \{\exp(a_x) \times E^*[\exp(b_x k_t)]\}, \\
 &= \sum_x W_x \times \left\{ \exp \left(a_x + b_x k_0 + b_x t \left(\alpha^* - \frac{1}{2} \sigma^{*2} - \lambda^* \gamma^* \right) \right. \right. \\
 &\quad \left. \left. + \frac{1}{2} b_x^2 \sigma^{*2} t + \lambda^* t \left(\frac{p^* \eta_1^*}{\eta_1^* - b_x} + \frac{q^* \eta_2^*}{\eta_2^* + b_x} - 1 \right) \right) \right\}.
 \end{aligned}$$

In our context, the fixed rate q_{fixed} is represented as $E^*[\mu_t]$, the expected future mortality rate in the risk neutral measure with W_x being a weight associated with the age category. These weights are given in Table 5.

The fixed rate or equivalently the mortality forward rate quoted by an investment bank would be formed using a combination of (1) best estimate mortality projection, (2) a risk premium, and (3) mid-to-bid spread (half of the ask-bid spread).

The best estimate of mortality will depend on the model used (e.g., the Lee-Carter model), and so may be biased if the model is not the most appropriate one. The risk premium must also be appropriate for the market and based on transactions

² Of course investors can take either side of the transaction, they do not need to go long or short as described above, only take the side they view as advantageous.

TABLE 5
Parameters for the Closed-Form Solution of the q-Forward

Age-Specific Parameters	W_x	a_x	b_x	Other Parameters	Value
<1	0.013818	-3.4087	0.1455	k_0	-10.302
1-4	0.055317	-6.2254	0.1960	t	10
5-14	0.145565	-7.1976	0.1492	α^*	-0.20
15-24	0.138646	-6.2957	0.0994	σ^*	0.31
25-34	0.135573	-5.9923	0.1044	λ^*	0.029
35-44	0.162613	-5.4819	0.0855	γ^*	-1.25
45-54	0.134834	-4.7799	0.0608	p^*	0.035
55-64	0.087247	-4.0137	0.0468	η_1^*	0.89
65-74	0.066037	-3.2347	0.0426	q^*	0.065
75-84	0.044842	-2.4196	0.0409	η_2^*	0.93
>85	0.015508	-1.6119	0.0290		

Note: We apply the risk-neutral measure change on the positive jump severity η_1^* and negative jump severity η_2^* .

there. Unfortunately to date, there are very few transactions involving mortality based derivatives and so the risk premium calculation in this market is currently problematic.

The calculation presented here uses the well-known risk neutral valuation approach with adjustments for mortality and longevity jumps. The jump processes play a role in fitting the data and in estimating the risk premium. The mortality forward rate flows from the closed-form solution. For the U.S. data used here, the fixed rate 10-year q-forward contract is priced at 0.8765 percent (or 87.65 basis points). This DEJD pricing may differ from that of the Lee-Carter or Chen-Cox models.

The Lee-Carter model, the Chen-Cox model and our DEJD model will naturally yield different values for the best estimate mortality projection. Not incorporating the asymmetric jump effect will make a difference in the calculated expected mortality rate that enters into the determination of the fixed rate payment component of the swap in Figure 6. Using the same data, whole age groups, and both sexes in the reference year for the U.S. national population, the result for the best estimate mortality projection for q-forward fixed rate is 0.8583 percent using the Lee-Carter model, 0.8594 percent using the Chen-Cox model, and 0.8765 percent using the DEJD model. Thus, the DEJD model will yield a larger expected fixed payment affecting the size of the experienced swap.

The Lee-Carter model captures the significant positive jump (1918 flu) and moderate negative jump in the baseline Brownian motion process, which caused a larger standard deviation and a positively biased mean for the baseline Brownian motion process. If we set the Lee-Carter model as the benchmark and compare the Chen-Cox model and the DEJD models, we conclude the following.

Compared to the Lee-Carter model, the Chen-Cox model considers the significant positive jump and moderate negative jump in the normal jump process, not in the

baseline Brownian motion process. This reduces the standard deviation and the mean in the baseline Brownian motion process. However, in the projection of future mortality rates, the jumps are assumed to occur symmetrically in the jump diffusion model of Chen–Cox (2009). In this way, the overall jump effects offset each other (positive and negative). The jump diffusion does not affect the mean of the Brownian motion process very much, so the pricing of the fixed rate is not much different from that obtained with the Lee–Carter model.

Compared to the Lee–Carter model, our DEJD model considers the possibility of both positive jumps and negative jumps in the jump diffusion process, not in the baseline Brownian motion process. This also reduces the standard deviation and the mean in comparison to the baseline Brownian motion process. The difference of the DEJD model from the Chen–Cox model is that in the projection of the future mortality rate, the jumps in the DEJD model are assumed to occur asymmetrically following the historical rule of infrequent significant positive jumps and more frequent moderate negative jumps. In this way, the overall jump effect adds a positive increment to the mean of Brownian motion process, which causes the pricing of the fixed rate to be higher than that obtained with the Lee–Carter model.

Some final comments about the q -forward prices developed in this article are in order. A pricing difference in the q -forward also can arise from the use of a different implied market price of risk. Due to the lack of a sufficient number of mortality securities in the market, the implied market price of risk used in this article in conjunction with our model was estimated using only one product, that is, the Swiss Re mortality catastrophe bond. Previous literature involving annuities and other actively traded mortality-linked securities have shown how multiple data points can be employed to calibrate an improved implied market price of risk. Applying a different estimated implied market price of risk in the formula will cause a difference in the quoted q -forward fixed rate. Also, in realistic deals, the mid-bid spread (defined as the half of the bid–ask spread), and other factors including judgment calls can also cause a difference in the quoted fixed rate of q -forward.

CONCLUSION

This article proposed a quantitative model to price mortality-linked securities, and provided a possible approach to measuring and managing longevity/mortality risk. Marked improvement in life expectancy has attracted public attention to the financial consequences of longevity risk on pension plans, long-term care insurance, and Social Security solvency. Longevity risk can also seriously affect the asset and liability balance of the pension fund and annuity providers. On the other side of the longevity/mortality market, life insurers have begun to show increased concern about increased mortality risk caused, for example, by sudden influenza or other natural or man-made catastrophic sources. A series of mortality-linked securities, for example, longevity bond, mortality bond, and other types of securities have been issued to manage and transfer the risk. Additionally, the recent market involving life settlement securitization whose pricing depends on modeling the life expectancy of insurance policyholders has boomed. Hence, modeling and pricing mortality-linked securities is crucial to risk management, product innovation, and formation of a liquid intermediate market.

This article proposed a stochastic mortality model to capture the observed feature of the historical mortality rate process and used this model to price mortality-linked securities. The baseline component of our model incorporates the advantages of the Lee–Carter framework, which describes the main trend and regular dynamics of historical mortality rate and is able to adjust for the cohort effect. The jump diffusion component in our model applies the compound Poisson DEJD to extend the Lee–Carter model so as to describe longevity jumps (negative jumps) and mortality jumps (positive jumps) separately. Our model accommodates the different features of longevity jumps and mortality jumps. Hence, the model fits the mortality time-series increment distribution much better than the previous models and explains the distribution skewness effect. In addition, our proposed model has an advantage of mathematical tractability and the ability to obtain a closed-form solution for the standard securities, like the q -forward contract, whose price depends on the expected future mortality rate. Since the DEJD model has the concise specification and closed-form density function, the likelihood function for the parameters can be easily expressed. In this way, the small number of parameters and concise likelihood function facilitate the calibration and application of the model in practice.

We calibrated our model with the historical mortality rate data 1900–2004 from National Center for Health Statistics and we compared the prices of the q -forward fixed rate, calculated by the Lee–Carter model, the Chen–Cox model, and our DEJD model. We found that our mortality model fits this data better and that the q -forward fixed rate in the contracted swap used for hedging longevity risk is higher from our model.

As a caveat, however, should note that we use only 104 years of recent data, and the last 104 years may not be the same as the next 104 year (technological innovations are occurring more rapidly now than in the past). Moreover, 104 years may not be adequate for modeling certain low frequency events. Can a 1918 flu epidemic scale event be expected once every 100 years or once every 200 years? Also, the fact that large longevity jumps have not occurred within this data set does not mean they have not occurred in the past (e.g., the acceptance of the germ theory of disease prior to this data set) or that they will not occur in the future. Cox, Lin, and Pedersen (2010) note that some experts conjecture that we may experience extreme events of both types in the future. As with all actuarial modeling, we have to accept that there is a real risk that the coefficients we estimate may prove to be incorrect if the past data are not reflective of the future. Model risk is always an actuarial issue, and our mortality model is no exception. For the creation of capital market financial instruments to hedge longevity risk such as in Figure 6, however, the model risk does not affect the hedger (the pension fund's hedge still works and the life insurer's hedge still works), and the investment bank intermediary is protected as long as they did not take a long or short position. Thus, while model risk is an important concern as far as mortality modeling is concerned, it is of less concern in the dual hedging context of this article.

REFERENCES

- American Council of Life Insurers, 2010, *Life Insurers Fact Book*. World Wide Web: <http://www.acli.com/Tools/Industry%20Facts/Life%20Insurers%20Fact%20Book/Pages/Default.aspx>.

- Anderson, J., 2009, Back to Business-Wall Street Pursues Profit in Bundles of Life Insurance, *New York Times*, September 5.
- Bauer, D., M. Borger, and J. Russ, 2010, On the Pricing of Longevity-Linked Securities, *Insurance: Mathematics and Economics*, 46: 139-149.
- Biffis, E., 2005, Affine Processes for Dynamic Mortality and Actuarial Valuation, *Insurance: Mathematics and Economics*, 37: 443-468.
- Blake, D., A. J. Cairns, and K. Dowd, 2006a, A Two-Factor Model for Stochastic Mortality With Parameter Uncertainty: Theory and Calibration, *Journal of Risk and Insurance*, 73(4): 687-718.
- Blake, D., A. J. Cairns, and K. Dowd, 2006b, Pricing Death: Frameworks for the Valuation and Securitization of Mortality Risk, *ASTIN Bulletin*, 36: 79-120.
- Blake, D., A. J. Cairns, K. Dowd, and R. MacMinn, 2006, Longevity Bonds: Financial Engineering, Valuation and Hedging, *Journal of Risk and Insurance*, 73(4): 647-672.
- Brouhns, N., M. Denuit, and J. K. Vermunt, 2002, A Poisson Log-Bilinear Regression Approach to the Construction of Projected Lifetables, *Insurance: Mathematics and Economics* 31: 373-393.
- Chen, H., and S. H. Cox, 2009, Modeling Mortality With Jumps: Application to Mortality Securitization, *Journal of Risk and Insurance*, 76(3): 727-751.
- Coughlan, G., D. Epstein, A. Ong, A. Sinha, J. Hevia-Portocarrero, E. Gingrich, M. Khalaf-Allah, and P. Joseph, 2007, Lifemetrics: A Toolkit for Measuring and Managing Longevity and Mortality Risks, Technical Report, J.P. Morgan.
- Coughlan, G., D. Epstein, A. Sinha, and P. Honig, 2007, q-Forwards: Derivatives for Transferring Longevity and Mortality Risk, Technical Report, J.P. Morgan.
- Cox, S. H., Y. Lin, and H. Pedersen, 2010, Mortality Risk Modeling: Application to Insurance Securitization, *Insurance: Mathematics and Economics*, 46: 242-253.
- Cox, S. H., Y. Lin, and S. Wang, 2006, Multivariate Exponential Tilting and Pricing Implications for Mortality Securitization, *Journal of Risk and Insurance*, 73: 719-736.
- Cummins, J. D., 2005, Securitization of Life Insurance Assets and Liabilities, *Journal of Risk and Insurance*, 72(2): 193-226.
- Cummins, J. D. and P. Trainar, 2009, Securitization, Insurance, and Reinsurance, *Journal of Risk and Insurance*, 76(3): 463-492.
- Dahl, M., 2004, Stochastic Mortality in Life Insurance: Market Reserves and Mortality-Linked Insurance Contracts, *Insurance: Mathematics and Economics*, 35: 113-136.
- Denuit, M., P. Devolder, and A. Goderniaux., 2007, Securitization of Longevity Risk: Pricing Survivor Bonds With Wang Transform in the Lee-Carter Framework, *Journal of Risk and Insurance*, 74(1): 87-113.
- Guterman, S., C. England, A. Parikh, and R. Pokorski, 2002, Living to 100 and Beyond: Implications for Retirement, *Record of Society of Actuaries*, 28(3): 135-164.
- Halonon, D., 2007, IRS Sets Costly Table [Electronic Version]. *Pension and Investments*, January 22. World Wide Web: <http://www.highbeam.com/doc/1G1-158440189.html>.
- Iacovino, J. R., 2009, Improvements in Life Expectancy: Past, Present and Future. Presented at the 5th Longevity Risk Conference, New York, New York,

- September 26. World Wide Web: <http://www.longevity-risk.org/five/presentations.html>.
- Jardine Lloyd Thompson, 2006, Pension Capital Strategies Releases Report on FTSE 100 Pension Disclosures, Pension Capital Strategies. World Wide Web: http://www.jltgroup.com/content/UK/employee_benefits/pressreleases/070117PCSQuaReportPR.pdf.
- Kou, S. G., and H. Wang, 2004, Option Pricing Under a Double Exponential Jump Diffusion Model, *Management Science*, 50(9): 1178-1192.
- Lee, R. D., and L. R. Carter, 1992, Modeling and Forecasting U.S. Mortality, *Journal of the American Statistical Association*, 87: 659-675.
- Li, S. H., and W. S. Chan, 2005, Outlier Analysis and Mortality Forecasting: The United Kingdom and Scandinavian Countries, *Scandinavian Actuarial Journal*, 2005(3): 187-211.
- Li, S. H., and W. S. Chan, 2007, The Lee-Carter Model for Forecasting Mortality, Revisited, *North American Actuarial Journal*, 11(1): 68-89.
- Lin, Y., and S. H. Cox, 2005, Securitization of Mortality Risks in Life Annuities, *Journal of Risk and Insurance*, 72: 227-252.
- Lin, Y., and S. H. Cox, 2008, Securitization of Catastrophe Mortality Risks, *Insurance: Mathematics and Economics*, 42: 628-637.
- Lin, Y., S. H. Cox, and H. Pedersen, 2010, Mortality Risk Modeling: Application to Insurance Securitization, *Insurance: Mathematics and Economics*, 46: 242-253.
- MacMinn, R. D., P. L. Brockett, and D. Blake, 2006, Longevity Risk and Capital Market Solutions, *Journal of Risk and Insurance*, 73(4): 551-557.
- Malcolm, L. J., V. L. Bengtson, P. G. Coleman, and T. B. L. Kirkwood, 2005, *The Cambridge Handbook of Age and Ageing* (Cambridge, UK: Cambridge University Press), p. 109.
- Milevsky, M. A., and S. D. Promislow, 2001, Mortality Derivatives and the Option to Annuitize, *Insurance: Mathematics and Economics*, 29: 299-318.
- Modu, E., 2009, Criteria—Insurance-Linked Securities, A.M. Best Methodology, November 24. World Wide Web: <http://www.ambest.com/ratings/methodology>.
- Ramezani, C. A., and Y. Zeng, 2007, Maximum Likelihood Estimation of the Double Exponential Jump-Diffusion Process, *Annals of Finance*, 3: 487-507.
- Renshaw, A. E., and S. Haberman, 2003, On the Forecasting of Mortality Reduction Factors, *Insurance: Mathematics and Economics*, 32: 379-401.
- Reuters News Service, 2010, UK Pension Liabilities Spike as People Live Longer, November 23. World Wide Web: <http://www.reuters.com/article/idUSLNE6AO01120101125>.
- Smithson, D., 2006, Improvements in Mortality and the Impact on Company Pension Plans, Mercer Human Resource Consulting. World Wide Web: <http://www.mercer.com/referencecontent.htm?idContent=1244560>.
- Toole, J., 2007, May, Potential Impact of Pandemic Influenza on the U.S. Life Insurance Industry, Research report, *Society of Actuaries*.
- Zanjani, G., 2002, Pricing and Capital Allocation in Catastrophe Insurance. *Journal of Financial Economics*, 65: 283-305.

Copyright of Journal of Risk & Insurance is the property of Wiley-Blackwell and its content may not be copied or emailed to multiple sites or posted to a listserv without the copyright holder's express written permission. However, users may print, download, or email articles for individual use.